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ON SOME PROPERTIES OF POLYNOMIALS.

BY E. L. DE FOREST.

(Continued from page 82.)

THE following property of polynomials of one variable deserves somewhat fuller treatment than we have yet given it. Suppose that the unlimited series

$$\dots u_{-2}, u_{-1}, u_0, u_1, u_2, \dots$$

is of algebraic form and of the r order, its terms being equidistant values of the function

$$u = A_0 + A_1x + A_2x^2 + \dots + A_rx^r, \quad (52)$$

and therefore ordinates to the curve of which this is the equation, the constant interval between them being $4x$. Supposing these terms to be adjusted by an unsymmetrical formula,

$$u'_0 = l_0 u_0 + l_1 u_1 + l_2 u_2 + \dots + l_m u_m + l_{-1} u_{-1} + l_{-2} u_{-2} + \dots + l_{-m} u_{-m}, \quad (53)$$

let us inquire what conditions must be satisfied by the coefficients l , in order that the adjusted term u'_0 may be an ordinate to the curve (52), though not necessarily the same as u_0 . Let $\mu \Delta x$ be the interval, if any, between u'_0 and u_0 , μ being a number, integral or fractional. Taking the position of u'_0 as origin of coordinates, μ is essentially + or — according as u_0 lies to the right or left of u'_0 . Hence:

$$\begin{aligned} u_0 &= A_0 + A_1 \mu \Delta x + A_2 \mu^2 (\Delta x)^2 + \dots + A_r \mu^r (\Delta x)^r, \\ u_1 &= A_0 + A_1 (\mu+1) \Delta x + A_2 (\mu+1)^2 (\Delta x)^2 + \dots + A_r (\mu+1)^r (\Delta x)^r, \\ u_{-1} &= A_0 + A_1 (\mu-1) \Delta x + A_2 (\mu-1)^2 (\Delta x)^2 + \dots + A_r (\mu-1)^r (\Delta x)^r, \\ &\vdots \\ u_m &= A_0 + A_1 (\mu+m) \Delta x + A_2 (\mu+m)^2 (\Delta x)^2 + \dots + A_r (\mu+m)^r (\Delta x)^r, \\ u_{-m} &= A_0 + A_1 (\mu-m) \Delta x + A_2 (\mu-m)^2 (\Delta x)^2 + \dots + A_r (\mu-m)^r (\Delta x)^r. \end{aligned}$$

Substitute these values in (53), and write for brevity

$$\left. \begin{aligned} b_0 &= l_0 + l_1 + l_{-1} + l_2 + l_{-2} + \dots + l_m + l_{-m}, \\ b_1 &= \mu l_0 + (\mu+1)l_1 + (\mu-1)l_{-1} + \dots + (\mu+m)l_m + (\mu-m)l_{-m}, \\ b_2 &= \mu^2 l_0 + (\mu+1)^2 l_1 + (\mu-1)^2 l_{-1} + \dots + (\mu+m)^2 l_m + (\mu-m)^2 l_{-m}, \\ b_r &= \mu^r l_0 + (\mu+1)^r l_1 + (\mu-1)^r l_{-1} + \dots + (\mu+m)^r l_m + (\mu-m)^r l_{-m}, \end{aligned} \right\} \quad (54)$$

and it will be found that (53) becomes

$$u'_0 = A_0 b_0 + A_1 b_1 \Delta x + A_2 b_2 (\Delta x)^2 + \dots + A_r b_r (\Delta x)^r. \quad (55)$$

This adjusted value u'_0 is the same value which u in (52) has at the origin, namely A_0 . But A_0 , A_1 , A_2 , &c., in (52), may be any arbitrary numbers, so that to make the second member of (55) always equal to A_0 , we must have

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = 0, \dots b_r = 0. \quad (56)$$

These are the conditions which the coefficients l must fulfil. The first one requires that the sum of all the l 's shall be unity. The second one determines μ , giving

$$-\mu = l_1 - l_{-1} + 2(l_2 - l_{-2}) + \dots + m(l_m - l_{-m}), \quad (57)$$

so that $-\mu \Delta x$ is the lever arm of the system of parallel forces l , about the place of l_0 . But since it is also the distance from l_0 or u_0 to the place of u'_0 , we see that the adjusted term u'_0 is located at the centre of parallel forces of the coefficients l . In the third condition, $b_2 (\Delta x)^2$ being the sum of the products formed by multiplying each l into the square of its distance from the centre of forces, $b_2 = 0$ requires that the radius of gyration of the coefficients about their centre of forces, shall be equal to zero. We use the term radius of gyration, as before, in an extended sense, for the l may be either positive or negative, and indeed some of them must be negative to make the radius zero. In general, $b_n = 0$ reduces to zero the sum of the products formed by multiplying each l into the n th power of its abscissa reckoned from the centre of forces as an origin. Now if (53) is so constructed as to satisfy these conditions, any term in the adjusted series is separated by the constant interval $\mu \Delta x$ from the place of that term of the given series which had the coefficient l_0 when the adjusted term in question was computed, wherefore the adjusted terms are equidistant, and being ordinates to the curve (52), they form a series of the r order. Let this series be re-adjusted by a second formula like that in (4), and let its coefficients L satisfy conditions similar to the foregoing, so that it gives another adjusted series of the r order, whose terms are equidistant, the adjusted term u''_0 being coincident with the centre of forces, and separated from the place of u'_0 by an interval $\mu' \Delta x$ which is constant, being the distance which separates L_0 from the centre of forces of this system. But as we saw in connection with (4),

the two adjustments are equivalent to a single adjustment made by a resultant formula, whose coefficients are those of the powers of z in the product of the two polynomials (5), these two having the same coefficients l and L which the component adjustment formulas (4) had. Denoting the coefficients in the resultant formula by λ , we see that since this formula gives an adjustment of the original series like that which (53) gives, only having the interval $(\mu + \mu')\Delta x$ in the place of $\mu\Delta x$, it must satisfy similar conditions (56), so that the sum of all the λ 's is unity, and the sum of the products of each λ into the n th power of its abscissa reckoned from the centre of forces in this system, is = zero, n being any integer from 1 to r . The l and L are numbers which may have any values, subject only to the conditions named. Any equation $b_n = 0$, or, as (54) gives it,

$$\mu^n l_0 + (\mu+1)^n l_1 + (\mu-1)^n l_{-1} + \dots + (\mu+m)^n l_m + (\mu-m)^n l_{-m} = 0,$$

will still hold good when each l in it is multiplied by a constant number. Any polynomial is equal to the product of a constant number into a polynomial the sum of whose coefficients is unity. Hence, neglecting the condition $b_0 = 1$, we have proved that if the remaining conditions are satisfied by the coefficients in each of two polynomials, such as (17), they will also be satisfied by the coefficients in their product. Consequently, if any number of polynomials, each satisfying these conditions, are multiplied together, their final product will satisfy the same conditions. If the conditions are satisfied by any one polynomial, they will also be satisfied by any power of that polynomial. This is the case already illustrated (ANALYST, Jan. 1880, p. 4), where the polynomial used as an example was

$$\frac{1}{81}(-4 + 30z + 60z^2 - 5z^3).$$

The divisor 81 may as well be omitted, dispensing with the condition $b_0 = 1$. As a further illustration, let us take the two polynomials

$$-4 + 30z + 60z^2 - 5z^3, \quad 1 - 4z + 6z^2 + 2z^3 + z^4,$$

the centre of forces in the first one being $\frac{2}{3}$ of the way from the 30 to the 60, while in the second one it coincides with the 2. In the first, we have both for $n = 2$ and $n = 3$,

$$b_n = -4(-\frac{5}{3})^n + 30(-\frac{2}{3})^n + 60(\frac{1}{3})^n - 5(\frac{4}{3})^n = 0;$$

and likewise in the second,

$$b_n = 1(-3)^n - 4(-2)^n + 6(-1)^n + 2(0)^n + 1(1)^n = 0.$$

According to our theory, the same two conditions, $b_2 = 0$ and $b_3 = 0$, ought to be satisfied by the product of these polynomials, which is

$$-4 + 46z - 84z^2 - 73z^3 + 436z^4 + 120z^5 + 50z^6 - 5z^7.$$

Its centre of forces is $\frac{2}{3}$ of the way from the 436 to the 120, and we find actually, both for $n = 2$ and $n = 3$,

$$b_n = -4(-\frac{14}{3})^n + 46(-\frac{11}{3})^n - 84(-\frac{8}{3})^n - 73(-\frac{5}{3})^n + 436(-\frac{2}{3})^n + 120(\frac{1}{3})^n + 50(\frac{4}{3})^n - 5(\frac{7}{3})^n = 0.$$

Polynomials of two variables possess properties analogous to the foregoing. Suppose that an unlimited double series, such as (6), is of algebraic form, its terms being values of the function

$$u = \frac{A_{0,0}x^0y^0|A_{r,s}x^ry^s}{A_{0,0}x^0y^0|A_{r,s}x^ry^s}.$$

For greater convenience of notation, we will express such a function by means of only two terms diagonally opposite, thus,

$$u = A_{0,0}x^0y^0||A_{r,s}x^ry^s. \quad (58)$$

The terms of the double series are ordinates to the surface of which (58) is the equation, the constant intervals between them being Δx and Δy . If these terms are to be adjusted by an unsymmetrical formula like the first one in (11), which we write more compactly thus,

$$u'_{0,0} = l_{-m,-n}u_{-m,-n}||l_{m,n}u_{m,n}, \quad (59)$$

we will inquire what conditions must be satisfied by the coefficients l , in order that the adjusted term $u'_{0,0}$ shall be also an ordinate to the surface, and be separated from $u_{0,0}$ by the constant horizontal and vertical intervals $\mu\Delta x$ and $\nu\Delta y$. Taking the position of $u'_{0,0}$ as the origin, the coordinates of the term $u_{m,n}$ will be $x = (\mu+m)\Delta x$, $y = (\nu+n)\Delta y$, so that (58) gives

$$u_{m,n} = A_{0,0}(\mu+m)^0(\nu+n)^0(\Delta x)^0(\Delta y)^0||A_{r,s}(\mu+m)^r(\nu+n)^s(\Delta x)^r(\Delta y)^s, \quad (60)$$

which is a general expression for any term in the given series. For example, $u_{2,-1}$ is expressed by putting $m=2$ and $n=-1$. If (60) is multiplied throughout by $l_{m,n}$, it becomes a general expression for any product, $l_{m,n}u_{m,n}$ in the adjustment formula (59). Supposing these expressions to be substituted for all the products in (59), it is evident that any const. independent of m and n , such as $A_{r,s}(\Delta x)^r(\Delta y)^s$ for example, will enter (59) with a coefficient which is the sum of all the values of

$$(\mu+m)^r(\nu+n)^s l_{m,n}$$

which can be formed by giving to m any integral value from $-m$ up to m , and to n any value from $-n$ to n . Denoting this sum by $b_{r,s}$, we have,

$$b_{r,s} = (\mu-m)^r(\nu-n)^s l_{-m,-n}||(\mu+m)^r(\nu+n)^s l_{m,n}, \quad (61)$$

and the adjusted term will stand

$$u'_{0,0} = A_{0,0}b_{0,0}(\Delta x)^0(\Delta y)^0||A_{r,s}b_{r,s}(\Delta x)^r(\Delta y)^s. \quad (62)$$

This is the same value which u in (58) has at the origin, namely $A_{0,0}$, and since the A 's in (58) are supposed to be arbitrary numbers given in advance, the second member of (62) will not in general be equal to $A_{0,0}$ unless we have

$$b_{0,0} = 1, \quad b_{r,s} = 0. \quad (63)$$

This last is to be understood as equating to zero all the values of $b_{r,s}$ in

(61) which can be formed by assigning to r and s all the pairs of values which the exponents of x and y have in (58), excepting only the pair $r=0$, $s=0$. The condition $b_{0,0}=1$ requires that the sum of all the l 's shall be unity. The two next conditions, $b_{1,0}=0$ and $b_{0,1}=0$, determine the relations of the l to μ and ν ;

$$-\mu = -ml_{-m,-n} || m l_{m,n}, \quad -\nu = -nl_{-m,-n} || n l_{m,n}, \quad (64)$$

so that $-\mu \Delta x$ is the lever arm of the system of parallel forces l about a vertical axis passing through the position of $l_{0,0}$, and $-\nu \Delta y$ is the lever arm about a horizontal axis through the same point. But these are also the horizontal and vertical distances from $l_{0,0}$ or $u_{0,0}$ to $u'_{0,0}$, whence it appears that the adjusted term $u'_{0,0}$ is located at the centre of parallel forces of the coefficients l . In general, any one condition $b_{r,s}=0$ reduces to zero the sum of the products formed by multiplying each l into the r power of its abscissa and the s power of its ordinate, reckoned from the centre of forces as an origin. By reasoning quite similar to that already followed in the case of a single series, and which need not be repeated here in detail, we get the following results.

Suppose that (59) satisfies the conditions (63), and that the adjusted terms, forming a series of the same order as the given series, are readjusted by a second formula like that in (11), with coefficients L satisfying conditions similar to those which the l satisfied, μ' and ν' taking the places of μ and ν , so that we have a second adjusted series of the same order. The adjusted term $u''_{0,0}$ will be located at the centre of forces for the L , and separated from $u'_{0,0}$ by horizontal and vertical intervals $\mu' \Delta x$ and $\nu' \Delta y$. The two adjustments are equivalent to a single one made by a resultant formula, whose coefficients λ are those of the powers of x and y in the product of the two polynomials (12), these two having the same coefficients l and L which the component formulas had. The λ will satisfy the same conditions (63) as the l and L did, and the adjusted term $u''_{0,0}$ will coincide with the centre of forces for λ , and will be separated from $u_{0,0}$ by the intervals $(\mu + \mu') \times \Delta x$ and $(\nu + \nu') \Delta y$. All of the conditions (63) except $b_{0,0}=1$ will hold good when the coefficients, as for instance the l 's in (61) are all multiplied by a constant number. Neglecting then the $b_{0,0}=1$, if the remaining conditions are satisfied by the coefficients in each of the two polynomials in x and y , they will also be satisfied by the coefficients in their product. If therefore several polynomials, all satisfying those conditions, are multiplied together, their final product will satisfy the same conditions.

If the conditions are satisfied by one polynomial, they will be satisfied by any power of it. We will illustrate these properties by a single example.

Suppose that the equation of the surface is

$$u = \left\{ \begin{array}{l} +A_{0,2} y^2 \\ +A_{0,1} y + A_{1,1} xy + A_{2,1} x^2 y \\ A_{0,0} + A_{1,0} x + A_{2,0} x^2 \end{array} \right\}, \quad (65)$$

its form being such that a transfer of the origin to any point in the plane of XY will not introduce any new term into the equation. The conditions to be satisfied by the l , omitting $b_{0,0} = 1$, are therefore

$$b_{1,0} = 0, b_{0,1} = 0, b_{1,1} = 0, b_{2,0} = 0, b_{0,2} = 0, b_{2,1} = 0, \quad (66)$$

the sub-indices of b being the same as the exponents of x and y . Two polynomials, whose coefficients are given in (67), have been constructed so that the coefficients of each will satisfy all these conditions. The coordinates

0	0	—2	— 6	12	18	6
3	6	3	—12	33	—18	57
3	6	—3	18	—55	60	—33

(67)

of their centres of forces, reckoned from the lower left hand coefficient as an origin, are made $x' = \frac{1}{2}4x$ and $y' = \frac{1}{2}4y$ in the first polynomial, and $x' = \frac{5}{2}4x$ and $y' = \frac{3}{2}4y$ in the second. In the first one, with the centre of forces as an origin, $b_{m,n}$ cleared of fractions is expressed as in (68), and it

$$2^{m+n} b_{m,n} = \left\{ \begin{array}{l} 0(-1)^m \binom{3}{n} + 0(1)^m \binom{3}{n} - 2(3)^m \binom{3}{n} \\ + 3(-1)^m \binom{1}{n} + 6(1)^m \binom{1}{n} + 3(3)^m \binom{1}{n} \\ + 3(-1)^m (-1)^n + 6(1)^m (-1)^n - 3(3)^m (-1)^n \end{array} \right\} \quad (68)$$

will be found to reduce to zero when m and n receive any one of those pairs of values which form the sub indices of b in (66). The coefficients of the second polynomial have a like property, their centre of forces being taken as the origin. Now multiply the two polynomials together, and the coefficients of the product stand as in (69). As already proved, the coordinates or lever arms of the centre of forces in the product will be the sum

0	0	12	—24	—36	—12
—18	0	132	96	126	—96
—54	27	216	362	150	219
18	—30	84	60	378	—270
54	—57	—204	426	—378	99

(69)

of those in the factors, namely $x' = (\frac{1}{2} + \frac{5}{2})\Delta x = 3\Delta x$, $y' = (\frac{1}{2} + \frac{3}{2})\Delta y = 2\Delta y$, so that this centre coincides with the 362. Taking it as the origin, we express $b_{m,n}$ as in (70), a part being omitted to save space. It will be found

$$b_{m,n} = \left\{ \begin{array}{l} 0(-3)^m 2^n + 0(-2)^m 2^n + 12(-1)^m 2^n - \&c. \\ -18(-3)^m 1^n + 0(-2)^m 1^n + 132(-1)^m 1^n + \&c. \\ -54(-3)^m 0^n + 27(-2)^m 0^n + 216(-1)^m 0^n + \&c. \\ + \dots \dots \dots \end{array} \right\} \quad (70)$$

that here for the product, as before for the two factors, $b_{m,n}$ becomes zero when m and n have any one of those pairs of values which are the sub-indices of b in (66). Wherever 0^0 occurs in (70) it has the value unity. That this must be so, is evident because the condition $b_{0,0} = 1$ signifies that the sum of all the coefficients is unity, and if this condition is satisfied by each of two factors, it will be by their product. When one coefficient in this product happens, as in the case of (69), to coincide with the centre of forces, it will stand multiplied by $0^m 0^n$ in (70), and other coefficients in the same row or column will be multiplied by 0^n or 0^m . When m and n are zero, the coefficients in question must still be counted in forming the sum $b_{0,0} = 1$, and this cannot be done unless 0^0 is here equal to unity.

Let us now take an algebraic function of three variables, x , y and z , whose terms are supposed to be arranged in the form of a rectangular block, as in the demonstration of (51). We place the origin at one corner occupied by an initial term $A_{0,0,0}$, while the opposite or farthest corner is occupied by the general term $A_{r,s,t} x^r y^s z^t$. For convenience of notation, we will express such a function by

$$u = A_{0,0,0} x^0 y^0 z^0 || A_{r,s,t} x^r y^s z^t. \quad (71)$$

When the variables are each supposed to take any equidistant values, increasing by Δx , Δy and Δz , the corresponding values of u form what may be called a triple series, any term of which may be taken as an initial term $u_{0,0,0}$, the position of the other terms being denoted by sub-indices reckoned from this in the three coordinate directions. Let any block of terms be adjusted by the unsymmetrical formula,

$$u'_{0,0,0} = l_{-m,-n,-p} u_{-m,-n,-p} || l_{m,n,p} u_{m,n,p}, \quad (72)$$

and we inquire what conditions must be satisfied by the coefficients l , in order that the adjusted term $u'_{0,0,0}$ may be always a value of the function (71), but separated in position from $u_{0,0,0}$ by the constant intervals $\mu\Delta x$, $\nu\Delta y$, and $\pi\Delta z$. Taking the origin at the adjusted term, the coordinates of $u_{m,n,p}$ will be $x = (\mu+m)\Delta x$, $y = (\nu+n)\Delta y$, $z = (\pi+p)\Delta z$, and (71) gives

$$u_{m,n,p} = A_{0,0,0} (\mu+m)^0 (\nu+n)^0 (\pi+p)^0 (\Delta x)^0 (\Delta y)^0 (\Delta z)^0 \\ || A_{r,s,t} (\mu+m)^r (\nu+n)^s (\pi+p)^t (\Delta x)^r (\Delta y)^s (\Delta z)^t, \quad (73)$$

a general expression for any term in the triple series. If it is multiplied throughout by $l_{m,n,p}$, it becomes an expression for any product $l_{m,n,p} u_{m,n,p}$ in the adjustment formula (72). Substituting such expressions for all the products in that formula, we see that any such quantity as

$$A_{r,s,t} (\Delta x)^r (\Delta y)^s (\Delta z)^t$$

will enter (72) with a coefficient which is the sum of all the values of

$$(\mu+m)^r (\nu+n)^s (\pi+p)^t l_{m,n,p}$$

which can be formed by giving to m any integral value from $-m$ to m , and to n any such value from $-n$ to n , and to p any such value from $-p$ to p . Denoting this sum by $b_{r,s,t}$, we have

$$b_{r,s,t} = (\mu-m)^r (\nu-n)^s (\pi-p)^t l_{-m,-n,-p} || (\mu+m)^r (\nu+n)^s (\pi+p)^t l_{m,n,p}, \quad (74)$$

and the adjusted term is expressed thus ;

$$u'_{0,0,0} = A_{0,0,0} b_{0,0,0} (\Delta x)^0 (\Delta y)^0 (\Delta z)^0 || A_{r,s,t} b_{r,s,t} (\Delta x)^r (\Delta y)^s (\Delta z)^t. \quad (75)$$

This is the same value which u in (71) has at the origin, namely $A_{0,0,0}$, and the second member of (75) will not in general = $A_{0,0,0}$ unless we have

$$b_{0,0,0} = 1, \quad b_{r,s,t} = 0, \quad (76)$$

which is to be understood as equating to zero all the values of $b_{r,s,t}$ in (74) which can be formed by giving to r, s and t all the sets of values which the exponents of x, y and z have in the several terms of (71), excepting only the set $r=0, s=0, t=0$. The condition $b_{0,0,0} = 1$ makes the sum of all the l 's equal to unity. The three next conditions,

$$b_{1,0,0} = 0, \quad b_{0,1,0} = 0, \quad b_{0,0,1} = 0,$$

determine the relations of the l to μ, ν and π , giving

$$\left. \begin{aligned} -\mu &= -ml_{-m,-n,-p} || ml_{m,n,p}, \\ -\nu &= -nl_{-m,-n,-p} || nl_{m,n,p}, \\ -\pi &= -pl_{-m,-n,-p} || pl_{m,n,p}. \end{aligned} \right\} \quad (77)$$

Thus $-\mu \Delta x, -\nu \Delta y$ and $-\pi \Delta z$ are the lever arms of the coefficients l with respect to three planes passing through the position of $l_{0,0,0}$ or $u_{0,0,0}$ and parallel to the coordinate planes. But since these lever arms are also the coordinate distances from $u_{0,0,0}$ to $u'_{0,0,0}$, it appears that the adjusted term is located at the centre of forces of the coefficients l . Any single condition $b_{r,s,t} = 0$ reduces to zero the sum of the products formed by multiplying each l into the r, s and t powers of its coordinates in the x, y and z directions respectively, reckoned from the centre of forces as an origin.

If the l in (72) satisfy the conditions (76), the adjusted triple series may be readjusted by another formula with coefficients L which satisfy the same conditions, only having μ', ν' and π' in the places of μ, ν and π . These two adjustments are equivalent to a single adjustment by a resultant

formula whose coefficients λ must satisfy the same conditions, μ, ν and π being here replaced by $(\mu + \mu'), (\nu + \nu')$ and $(\pi + \pi')$. By reasoning precisely analogous to that which we have already followed in the cases of single and double series, in connection with (4) and (11), it will appear that the resultant of two triple adjustment formulas such as (72), with coefficients l and L , is a formula in which any term $u_{r,s,t}$ of the given series enters with a coefficient λ which is the sum of all the products that can be formed by multiplying the L into the l so that in each product the sum of the first sub-indices shall be r , the sum of the second, shall be s , and the sum of the third, shall be t .

For example, denoting the two component formulas as in (72), by

$$u'_{0,0,0} = l_{-m,-n,-p} u_{-m,-n,-p} \parallel l_{m,n,p} u_{m,n,p},$$

$$u''_{0,0,0} = L_{-h,-i,-j} u'_{-h,-i,-j} \parallel L_{h,i,j} u'_{h,i,j},$$

it is evident that if a given term $u_{r,s,t}$ enters $u''_{0,0,0}$ through $u'_{0,0,0}$, it enters with the coefficient $L_{0,0,0} l_{r,s,t}$. If it also enters through

$$u'_{1,0,0}, \quad u'_{0,1,0}, \quad u'_{0,0,1}, \quad u'_{1,1,0}, \quad \&c.,$$

it has the respective coefficients

$$L_{1,0,0} l_{(r-1),s,t}, \quad L_{0,1,0} l_{r,(s-1),t}, \quad L_{0,0,1} l_{r,s,(t-1)}, \quad L_{1,1,0} l_{(r-1),(s-1),t},$$

and so on, each product Ll being such that the sums of the two first, the two second, and the two third sub-indices, are r, s and t respectively. The total resultant coefficient λ is the sum of all the products of the L into the l which can be formed in this way. The several values of λ will thus be identical with the coefficients in the product of the two polynomials

$$l_{-m,-n,-p} x^{-m} y^{-n} z^{-p} \parallel l_{m,n,p} x^m y^n z^p,$$

$$L_{-h,-i,-j} x^{-h} y^{-i} z^{-j} \parallel L_{h,i,j} x^h y^i z^j,$$

or what is the same thing, the coefficients in the product of

$$l_{-m,-n,-p} x^0 y^0 z^0 \parallel l_{m,n,p} x^{2m} y^{2n} z^{2p},$$

$$L_{-h,-i,-j} x^0 y^0 z^0 \parallel L_{h,i,j} x^{2h} y^{2i} z^{2j},$$

these two factors having coefficients which are the same and in the same order as the l and L of the two adjustment formulas.

All the conditions (76) except $b_{0,0,0} = 1$ will hold good when the l 's or L 's are multiplied by a constant number. Neglecting then the $b_{0,0,0} = 1$, if the remaining conditions are satisfied by the coefficients in each of two polynomials in x, y and z , they will also be satisfied by the coefficients in their product. If therefore a polynomial satisfying such conditions is raised to any power, or if several such polynomials are multiplied together, the final power or product will satisfy the same conditions. As regards the number of conditions (76) that may be applied, answering to the number of terms in the function (71), we remark that some of the highest terms and conditions may be absent, provided that those which remain form a complete

series as far as they go, so that if the origin is shifted by substituting $x+a$, $y+b$ and $z+c$ for x , y and z in (71), this will not introduce any terms of different degree from those already there. A similar remark applies in the cases of single and double series.

As an illustration of the foregoing, suppose that the function (71) has the form

$$u = A + Bx + Cy + Dz + Exz + Fy^2, \quad (78)$$

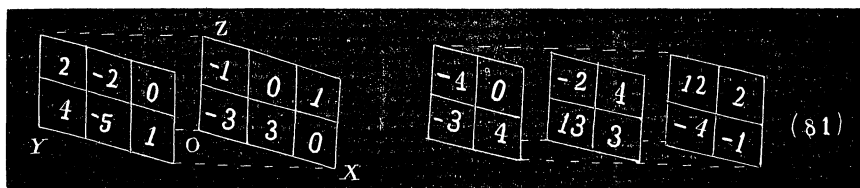
so that the conditions to be satisfied by the coefficients l , omitting $b_{0,0,0} = 1$, are

$$b_{1,0,0} = 0, b_{0,1,0} = 0, b_{0,0,1} = 0, b_{1,0,1} = 0, b_{0,2,0} = 0. \quad (79)$$

The two polynomials

$$\left. \begin{aligned} & -3 + 3x + 4y - z - 5xy + 2yz - 2xyz + x^2y + x^2z, \\ & -4 - x + 13y + 12z + 3xy + 2xz - 2yz + 4xyz - 3y^2 + 4xy^2 - 4y^2z, \end{aligned} \right\} \quad (80)$$

have been constructed so that their coefficients satisfy these conditions. Arranged in block form, they appear as in the diagram (81). The lines OX , OY , OZ give merely the directions, not the locations of the axes.



Starting from the first coefficient as an origin, that is, from the -3 in the one case and the -4 in the other, the centres of forces are placed in both cases at the point

$$x' = \frac{1}{2}\Delta x, \quad y' = \frac{1}{2}\Delta y, \quad z' = \frac{1}{2}\Delta z,$$

being at the middle of the block of eight coefficients nearest the origin. Making this centre of forces an origin, we have for any value of b in the first polynomial for instance, the expression (82), which reduces to zero for

$$2^{m+n+p}b_{m,n,p} = \left\{ \begin{aligned} & 2(-1)^m(1)^n(1)^p - 2(1)^m(1)^n(1)^p + 0 \\ & + 4(-1)^m(1)^n(-1)^p - 5(1)^m(1)^n(-1)^p + 1(3)^m(1)^n(-1)^p \\ & - 1(-1)^m(-1)^n(1)^p + 0 + 1(3)^m(-1)^n(1)^p \\ & - 3(-1)^m(-1)^n(-1)^p + 3(1)^m(-1)^n(-1)^p + 0 \end{aligned} \right\} \quad (82)$$

any set of values of m , n , p which form the sub-indices of b in (79). The second polynomial has a similar property. Now when the two polynomials are multiplied together, the coefficients in their product stand as in the diagram (83). As previously shown, the lever arms or coordinates x' , y' , z' of the centre of forces in the product are the sums of

(83)

-8	8	0	0
-22	34	-12	0
-12	31	-23	4

0	12	-12	0
33	-10	-31	8
61	-74	10	3

26	-24	-6	4
33	-67	29	5
-55	46	10	-1

-12	-2	12	2
-32	31	2	-1
12	-6	-3	0

those in the two factors, so that the centre of forces here coincides with the —67. Estimating $b_{m,n,p}$ from this centre as an origin, we shall find that it reduces to zero for the same sets of values of m, n, p as it did in the cases of the two factors.

SOLUTION OF A SPECIAL CASE OF PROBLEM 288, WHERE THE CIRCLES ARE TANGENT AT THE SAME POINT OF AB.

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First part. Geometrical Demonstration.—The triangle CMN being isosceles, $\angle CMN = \angle CNM$.
 $\therefore \angle NMD = \angle MND' = \angle NSD'$; $\therefore$ the triangles QMD and QSD are similar, and $MD : SD :: r : R :: QD : QD'$. By division, $(R-r) : R :: DD' : QD'$, or $(R-r) : R :: (R+r) : QD'$;
 $\therefore QD' = \frac{R(R+r)}{R-r}$,

i. e. QD' is constant, and MN in all positions will pass through Q . Similarly it may be shown that NM' in all positions will pass through the point Q' .

$$QO = \frac{2Rr}{R-r}, \quad Q'O = \frac{2Rr}{R+r}.$$

Analytical Demonstration.—Let O be the origin, QD' and BC the axes, (x_1, y_1) coordinates of M or M' , (x_2, y_2) coordinates of N , $(0, a)$ coordinates of the variable point C . Then we easily obtain

